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On the Chromatic Number of Cycle Books Graph

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ABSTRACT

Graph coloring is a fundamental topic in graph theory, with various applications in scheduling, networking, and optimization problems. In this study, we investigate the chromatic number of the cycle books graph $B_{C_n,m}$, a structured graph formed by attaching multiple cycles to a common path P_2. We establish that the chromatic number of $B_{C_n,m}$ depends on the parity of n. Specifically, we prove that if n is even, the chromatic number is $\chi(B_{C_n,m}) = 2$, while if n is odd, the chromatic number is $\chi(B_{C_n,m}) = 3$. These results provide a deeper understanding of coloring properties in book-like graphs and contribute to the broader study of chromatic numbers in structured graph families. The findings may be extended to other variations of book graphs and related topologies in future research.

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1. INTRODUCTION

The history of graph colouring originates from the Four-Color Problem, which was first proposed by Francis Guthrie in 1852. Guthrie, a student at University College London, observed that any map of regions in England could be colored using at most four colours, ensuring that no two adjacent regions shared the same color [1].

The map coloring problem remained formally unsolved until the 20th century. Eventually, Kenneth Appel and Wolfgang Haken rigorously proved the Four-Color Theorem in 1976 with the aid of a computer [2]. This proof was the first in the history of mathematics to rely heavily on computation, initially sparking controversy as it could not be manually verified by humans within a reasonable timeframe.

Graph coloring is an important and actively researched topic within graph theory, particularly in the context of vertex coloring. Vertex coloring of a graph G is defined as a mapping $c: V(G) \to \{1,2,3, \ldots, k\}$ assigning different labels (colors) to vertices so that no pair of adjacent vertices shares the same color (i.e., $c(u) \neq c(v)$ whenever vertices $u, v \in V(G)$ are adjacent). A graph G is said to have a k-coloring if exactly k distinct colors are used. The minimum number k for which a valid vertex coloring exists is known as the chromatic number of graph G, denoted by $\chi(G)$. Thus, the chromatic number $\chi(G)$ represents the minimal set size of colors required for proper vertex coloring.

Figure 1, the graph G_1 consists of three vertices, thereby implying $\chi(G1) \leq 3$. Similarly, graph G_2 has 4 vertices, resulting in $\chi(G2) \leq 4$. Since both G_1 and G_2 are complete graphs (every vertex is adjacent to all other vertices), we must also have $\chi(G1) \geq 3$ and $\chi(G2) \geq 4$. Consequently, it follows that $\chi(G1) = 3$and $\chi(G2) = 4$.

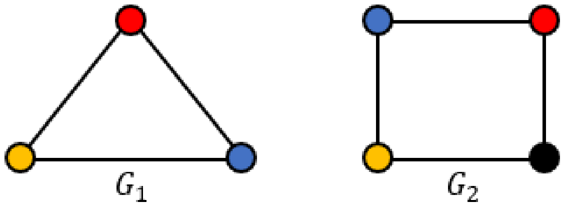

Figure 1. Vertex coloring.

Researching the chromatic number of graphs remains an interesting and popular topic in graph theory research. Several previous studies have examined the chromatic number in various contexts. Fundamental concepts of graph coloring and chromatic numbers were discussed in [3]. Dynamic coloring concepts in graphs and their implications for dynamic chromatic numbers were explored in [4]. In [5] investigated dynamic chromatic numbers across various classes of graphs and established related bounds. Reference [6] analyzed the corona product of graphs and its effects on the Gutman index and degree distance, both related to graph coloring. S. Jahanbekam [7] focused on r-dynamic coloring in graphs and determined the r-dynamic chromatic numbers for various graph types. The r-dynamic chromatic number and related bounds and characterizations for specific classes of graphs were discussed in [8]. Characteristics and coloring of wheel graphs and gear graphs were examined in [8]. Ainun *et al.* [9] have shown that addressed the Welch-Powell algorithm for graph coloring applications in course scheduling.

Further applications of the Welch-Powell algorithm for scheduling courses and assistants using graph-coloring concepts were explored in [10]. Abdy *et al.* [11] discussed constructing dual graphs of wheel graphs and determining their chromatic numbers. Kristiana *et al.* [12] studied the chromatic number of vertex-edge weighted coloring for corona products of path graphs with various other graphs. Recent research examining the chromatic number of graphs is as follows. Chromatic number of graphs specifically (P_5, HVN)-free graphs in [13], chromatic number of heptagraphs in [14], chromatic number of (n, m) graphs, (P_6, Diamond)-Free Graphs, and Locating-Chromatic Number of Graphs in [15], [16], and [17]. As discussed in [18], refined estimations for the chromatic number can be derived by examining specific properties of cycles in graphs. The structural properties of graphs containing multiple odd-length cycles have been explored in [19], providing insights into their chromatic complexity. The influence of different cycle lengths on the chromatic number has been extensively analyzed in [20], offering theoretical bounds relevant to this study.

A cycle graph is a simple undirected graph consisting of n vertices connected in a closed path, where each vertex has degree two. It is denoted by C_n, with vertex set $V = \{v_1, v_2, v_3, \dots, v_n\}$ and edge set $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), \dots, (v_{n-1}, v_n), (v_n, v_1)\}$. A book graph B_k is a graph formed by k triangles that share a common edge, resembling a book with k pages hinged at a single spine. Extending these concepts, a cycle book graph $B_{C_n,m}$ consists of m cycles of length n that share a common edge, combining the regularity of cycles with the layered structure of book graphs.

An equivalent definition of the cycle book graph was introduced in [21] as a graph consisting of m cycles sharing a common path P_2 . Prior studies have examined various parameters of this graph, such as metric and partition dimensions [22], and partition dimension specifically for the case $n = 3$ and $m \geq 4$ [23]. While the chromatic number of general book graphs has been previously examined, such as in [24], studies specifically focusing on the chromatic number of cycle book graphs remain limited. This research aims to investigate and characterize the chromatic number of cycle book graphs to enrich the current understanding of their coloring behavior.

2. RESEARCH METHOD

This research used a literature review as the research method from journals and research on chromatic number on graphs. The following provides the basic concepts needed in the construction of the concept of ideal. The cycle books graph is a graph consisting of m copies cycle C_n with the common path P_2.

2.1. Literature Review on Cycle Books Graph

The research begins with a comprehensive literature review focusing on graph coloring and, more specifically, on cycle book graphs, denoted by $B_{C_n,m}$. This step involves studying existing research, journals, books, and mathematical resources that discuss chromatic numbers and structural properties of cycle-based graphs. Through this review, the structure and characteristics of cycle book graphs are understood in preparation for deeper theoretical analysis.

To support the analysis, several theorems related to graph coloring are reviewed. These include: The chromatic number of cycle graphs, The Four Color Theorem, Brooks' Theorem, The properties of bipartite graphs and chromatic bounds. These theorems serve as foundational tools for reasoning about the chromatic behavior of cycle book graphs.

2.2. Formulation of Theorem

Based on the observations and analysis of the structure of $B_{C_n,m}$, a main theorem is proposed regarding its chromatic number. The theorem expresses the chromatic number $\chi\left(B_{C_n,m}\right)$ as a function of the parity of n (whether n is even or odd).

2.3. Determination of the Chromatic Number

To prove the theorem systematically, the problem is analyzed under two cases: when n is even number and when n is odd number. This division into cases allows for the application of appropriate coloring strategies based on the structural characteristics of the graph in each case, ensuring a more precise and well-directed proof.

After constructing valid coloring functions for both cases and verifying the correctness through adjacency conditions, the chromatic number of $B_{C_n,m}$ is determined: When n is even, two colors suffice ($\chi = 2$) and When n is odd, three colors are required ($\chi = 3$). These results confirm the theorem and complete the proof through theoretical analysis. The general flow of this research is outlined in the following diagram.

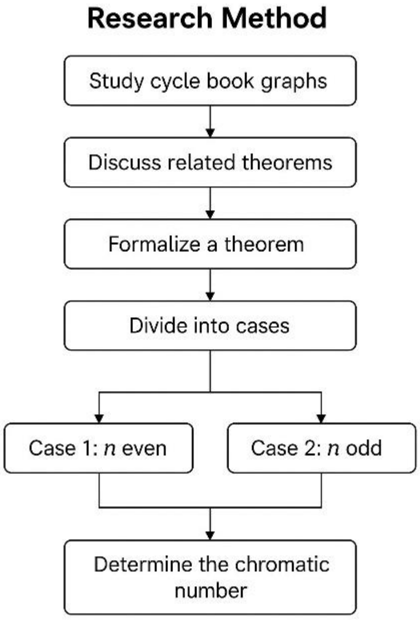

Figure 2. Research Method Scheme

3. RESULTS AND ANALYSIS

This section discusses the chromatic number of cycle books graphs. A cycle books graph consists of several copies of a cycle graph connected by a common path P_2. **Figure 3** illustrates graph $B_{C_3,m}$, which consists of multiple copies (specifically m copies) of cycle C_3 with a common path P_2.

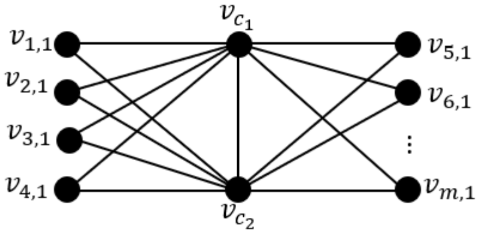

Figure 3. Graph $B_{C_3,m}$

The chromatic number $\chi(G)$ of a graph G is defined as the minimum number of colors required to color the vertices such that no two adjacent vertices have the same color. In this section, we determine the chromatic number of the cycle books graph, denoted as $\chi\left(B_{C_n,m}\right)$. To determine the chromatic number of cycle books graphs, several theorems related to the topic will be discussed. The following theorems will be used in proving the chromatic number of cycle books graphs.

Theorem 1. [25] A graph G has chromatic number $\chi(G) = 1$ if and only if it is an empty graph.

Theorem 2. (Brooks' Theorem) [26]. Let G be a connected undirected simple graph with maximum degree $\Delta(G)$. Then $\chi(G) \leq \Delta(G)$ unless G is a complete graph or an odd cycle, in which case: $\chi(G) = \Delta(G) + 1$.

Theorem 3. (The Four Color Theorem) [27]. Every map in a plane can be colored using no more than four colors, such that no two regions that share a common boundary segment (not just a point) have the same color. Formally, for any planar graph G, $\chi(G) \leq 4$.

In this section, we present the chromatic number of the cycle books graph, which is established in the following theorem.

Theorem 4 Let $B_{C_n,m}$ be a cycle books graph, where $n \geq 3$, and $n, m \in N$. The chromatic number of $B_{C_n,m}$ is given by:

$$\chi\left(B_{C_n,m}\right) = \begin{cases} 2, \text{if n is even,} \\ 3, \text{if n is odd.} \end{cases}$$

Proof.

The graph $B_{C_n,m}$ has an ordered vertex set given by $V = \{v_{c_2}, v_{c_1}\} \cup \{v_{1,1}, v_{1,2}, v_{1,3}, \dots, v_{1,n}\} \cup \{v_{2,1}, v_{2,2}, v_{2,3}, \dots, v_{2,n}\} \cup \{v_{3,1}, v_{3,2}, v_{3,3}, \dots, v_{3,n}\} \cup \{v_{m,1}, v_{m,2}, v_{m,3}, \dots, v_{m,n}\}$. The edge set E of the graph $B_{C_n,m}$ is ordered as follows:

$$\begin{aligned} &\{(v_{c_2}, v_{c_1})\} \cup \{(v_{c_1}, v_{1,1}), (v_{1,1}, v_{1,2}), (v_{1,2}, v_{1,3}), \dots, (v_{1,n-1}, v_{1,n}), (v_{1,n}, v_{c_2})\} \\ &\cup \{(v_{c_1}, v_{2,1}), (v_{2,1}, v_{2,2}), (v_{2,2}, v_{2,3}), \dots, (v_{2,n-1}, v_{2,n}), (v_{2,n}, v_{c_2})\} \\ &\cup \{(v_{c_1}, v_{3,1}), (v_{3,1}, v_{3,2}), (v_{3,2}, v_{3,3}), \dots, (v_{3,n-1}, v_{3,n}), (v_{3,n}, v_{c_2})\} \\ &\vdots \\ &\cup \{(v_{c_1}, v_{m,1}), (v_{m,1}, v_{m,2}), (v_{m,2}, v_{m,3}), \dots, (v_{m,n-1}, v_{m,n}), (v_{m,n}, v_{c_2})\} \end{aligned}$$

Suppose $B_{C_n,m}$ is a cycle books graph consisting of m cycles, each having a common edge, and each cycle containing n vertices, with conditions $n \geq 3$, $m \geq 2$, and $n, m \in N$. We divide the proof into two cases based on the parity of n.

Case 1. For n is even. To determine the chromatic number for this case, define the coloring function $c: V(v_{m,n}) \to \{1,2\}$ by assigning colors alternately to each vertex, according to the ordering of connected vertices, as follows.

$$\begin{aligned} &v_{c_2} \to 1, v_{c_1} \to 2, v_{1,1} \to 1, v_{1,2} \to 2, v_{1,3} \to 1, v_{1,4} \to 2, v_{1,5} \to 1, v_{1,6} \to 2, \dots, v_{1,n} \to 2, \\ &v_{c_2} \to 1, v_{c_1} \to 2, v_{2,1} \to 1, v_{2,2} \to 2, v_{2,3} \to 1, v_{2,4} \to 2, v_{2,5} \to 1, v_{2,6} \to 2, \dots, v_{2,n} \to 2, \\ &v_{c_2} \to 1, v_{c_1} \to 2, v_{3,1} \to 1, v_{3,2} \to 2, v_{3,3} \to 1, v_{3,4} \to 2, v_{3,5} \to 1, v_{3,6} \to 2, \dots, v_{3,n} \to 2, \\ &v_{c_2} \to 1, v_{c_1} \to 2, v_{m,1} \to 1, v_{m,2} \to 2, v_{m,3} \to 1, v_{m,4} \to 2, v_{m,5} \to 1, v_{m,6} \to 2, \dots, v_{m,n} \to 2. \end{aligned}$$

It can be observed that each vertex is adjacent only to vertices with different colors; thus, this coloring is valid. From the coloring function defined above, it is clear that two colors are sufficient to color this graph. Hence, we have established that $\chi(B_{C_n,m}) \leq 2$. To determine the lower bound of the chromatic number, we use **Theorem 1**, which states that a graph has chromatic number $\chi(G) = 1$ if and only if it is an empty graph. Clearly, the graph $B_{C_n,m}$ is not empty, implying that $\chi(B_{C_n,m}) \geq 2$. Thus, we conclude that for even n, $\chi(B_{C_n,m} = 2$. An illustrative example for this case, showing the chromatic number of the graph $B_{C_8,6}$, is presented in **Figure 4**.

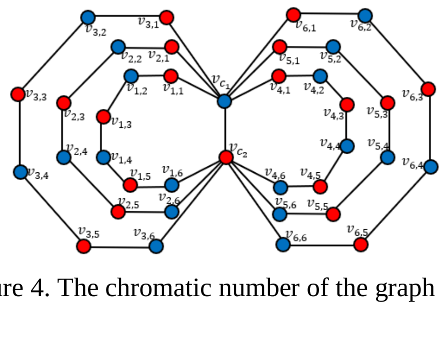

Figure 4. The chromatic number of the graph $B_{C_8,6}$

Case 2. For n is odd. Consider the graph $B_{C_n,m}$ with n vertices, where n is an odd integer. In this case, it will be shown that $\chi\left(B_{C_n,m}\right) = 3$. **Figure 5** provides an example showing the chromatic number of the graph $B_{C_7,6}$.

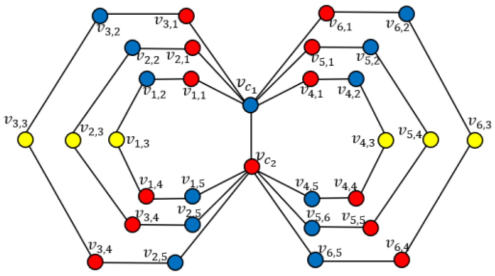

Figure 5. The chromatic number of the graph $B_{C_7,6}$

To prove this case, assume the coloring initially follows the same pattern as in Case 1, defining the coloring function $c: V\left(v_{m,n}\right) \rightarrow \{1,2\}$, assigning colors alternately to each vertex along the ordered connected vertices as follows.

$$v_{c_2} \rightarrow 1, v_{c_1} \rightarrow 2, v_{1,1} \rightarrow 1, v_{1,2} \rightarrow 2, v_{1,3} \rightarrow 1, v_{1,4} \rightarrow 2, v_{1,5} \rightarrow 1, v_{1,6} \rightarrow 2, \ldots, v_{1,n} \rightarrow 1,$$
$$v_{c_2} \rightarrow 1, v_{c_1} \rightarrow 2, v_{2,1} \rightarrow 1, v_{2,2} \rightarrow 2, v_{2,3} \rightarrow 1, v_{2,4} \rightarrow 2, v_{2,5} \rightarrow 1, v_{2,6} \rightarrow 2, \ldots, v_{2,n} \rightarrow 1,$$
$$v_{c_2} \rightarrow 1, v_{c_1} \rightarrow 2, v_{3,1} \rightarrow 1, v_{3,2} \rightarrow 2, v_{3,3} \rightarrow 1, v_{3,4} \rightarrow 2, v_{3,5} \rightarrow 1, v_{3,6} \rightarrow 2, \ldots, v_{3,n} \rightarrow 1,$$
$$\vdots$$
$$v_{c_2} \rightarrow 1, v_{c_1} \rightarrow 2, v_{m,1} \rightarrow 1, v_{m,2} \rightarrow 2, v_{m,3} \rightarrow 1, v_{m,4} \rightarrow 2, v_{m,5} \rightarrow 1, v_{m,6} \rightarrow 2, \ldots, v_{m,n} \rightarrow 1.$$

With the above coloring, since n is odd, after assigning colors up to vertex $n-1$, we return to the initial vertex v_{c_2}, which was initially colored 1. However, vertex $v_{m,n}$, which directly connects to v_{c_2}, is also colored 1, which contradicts the conditions required for a valid vertex coloring. To resolve this issue, a third color is necessary for the final vertex. Thus, vertex $v_{m,n}$ is assigned color 3, implying $\chi\left(B_{C_n,m}\right) \geq 3$. The revised vertex coloring function is defined as $c: V\left(v_{m,n}\right) \rightarrow \{1,2,3\}$, assigning colors alternately to each vertex along the ordered connected vertices as follows.

$$v_{c_2} \rightarrow 1, v_{c_1} \rightarrow 2, v_{1,1} \rightarrow 1, v_{1,2} \rightarrow 2, v_{1,3} \rightarrow 1, v_{1,4} \rightarrow 2, v_{1,5} \rightarrow 1, v_{1,6} \rightarrow 2, \ldots, v_{1,n} \rightarrow 3,$$
$$v_{c_2} \rightarrow 1, v_{c_1} \rightarrow 2, v_{2,1} \rightarrow 1, v_{2,2} \rightarrow 2, v_{2,3} \rightarrow 1, v_{2,4} \rightarrow 2, v_{2,5} \rightarrow 1, v_{2,6} \rightarrow 2, \ldots, v_{2,n} \rightarrow 3,$$
$$v_{c_2} \rightarrow 1, v_{c_1} \rightarrow 2, v_{3,1} \rightarrow 1, v_{3,2} \rightarrow 2, v_{3,3} \rightarrow 1, v_{3,4} \rightarrow 2, v_{3,5} \rightarrow 1, v_{3,6} \rightarrow 2, \ldots, v_{3,n} \rightarrow 3,$$
$$\vdots$$
$$v_{c_2} \rightarrow 1, v_{c_1} \rightarrow 2, v_{m,1} \rightarrow 1, v_{m,2} \rightarrow 2, v_{m,3} \rightarrow 1, v_{m,4} \rightarrow 2, v_{m,5} \rightarrow 1, v_{m,6} \rightarrow 2, \ldots, v_{m,n} \rightarrow 3.$$

This coloring is valid as it satisfies the vertex-coloring rules of the graph, ensuring each vertex is adjacent only to vertices of different colors. The coloring function clearly demonstrates that three colors suffice for this graph, thus establishing $\chi\left(B_{C_n,m}\right) \leq 3$. Consequently, we have proven that $\chi\left(B_{C_n,m}\right) = 3$ when is odd.

Based on these two cases, we conclude:

$$\chi\left(B_{C_n,m}\right) = \begin{cases} 2, \text{if n is even,} \\ 3, \text{if n is odd.} \end{cases}$$

This completes the proof.

4. CONCLUSION

In this study, we have determined the chromatic number of the cycle books graph $B_{C_n,m}$. Our findings establish that the chromatic number depends on the parity of n. Specifically, we have proven that:

- If n is even, the chromatic number $\chi(B_{C_n,m})$ is 2.
- If n is odd, the chromatic number $\chi(B_{C_n,m})$ is 3.

These results provide a clear characterization of the chromatic number for cycle books graphs, contributing to the broader understanding of graph coloring in structured graph families. Future work may explore generalizations to other book-like graphs or variations involving additional constraints.

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